Computers and Mathematics with Applications 59 (2010) 142–148

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Source term identification for an axisymmetric inverse heat conduction problem[☆]

Wei Cheng[a], Ling-Ling Zhao[b], Chu-Li Fu[c,*]

[a] College of Science, Henan University of Technology, Zhengzhou 450001, PR China

[b] Department of Mathematic and Information Science, Zhengzhou University of Light Industry, Zhengzhou 450002, PR China

[c] School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, PR China

ARTICLE INFO

Article history:
Received 15 September 2008
Received in revised form 19 August 2009
Accepted 19 August 2009

Keywords:
Ill-posed problem
Inverse source problem
Identification
Regularization
Error estimate

ABSTRACT

We consider an inverse heat source problem of determining the heat source term from the final temperature history of a cylinder. This problem is ill-posed. A simplified Tikhonov regularization method is applied to formulate regularized solution, which is stably convergent to the exact one with a logarithmic type error estimate.

1. Introduction

Inverse source problem is important in many branches of engineering sciences. For example, an accurate estimation of pollutant source is crucial to environmental safeguard in cities with high populations. So the problem has a long development history [1–6]. This problem is ill-posed: the solution (if it exists) does not depend continuously on measured data [7]. Some theories and effective algorithm have been obtained for the problem. For instance, the uniqueness and conditional stability results can be found in [8,9].

Taking more and more important role in migration of groundwater, identification and control of pollution source and environmental protection, the inverse source problem has been considered by many authors by different methods in recent years [10–17]. Such as mollification method [10,11], boundary element method [12], the finite difference method [13] and radial basis functions method [14].

However, the results available in the literature on the inverse heat source problem are mainly devoted to numerical algorithms. A few works of analytical methods were presented for this problem [18,19]. But in most cases the stability theory with explicitly error estimate has not been generalized accordingly. In this paper, we use a simplified Tikhonov regularization method to deal with an inverse heat source problem in an axisymmetric region and obtain a logarithmic type error estimate for the regularized solution.

The physical model considered here is an infinitely long cylinder of radius r_0 with initial temperature, and it is considered axisymmetric and surface temperature distribution holds zero [20]. The corresponding mathematical model of our problem

☆ The project is supported by the NNSF of China (No.10671085), the Fundamental Research Fund for Natural Science of Education Department of Henan Province of China (No. 2009B110007) and the Hight-level Personnel fund of Henan University of Technology (No. 2007BS028).

* Corresponding author.
E-mail addresses: chwei02@lzu.cn (W. Cheng), fuchuli@lzu.edu.cn (C.-L. Fu).

0898-1221/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2009.08.038

can be described by the following axisymmetric heat equation:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} = f(r), \quad 0 < r < r_0, 0 < t < T, \tag{1.1}$$

with the initial condition and final observations at T

$$u(r,0) = 0, \qquad u(r,T) = g(r), \quad 0 \le r \le r_0, \tag{1.2}$$

and the boundary conditions

$$u(r_0,t) = 0, \qquad \lim_{r\to 0} u(r,t) \text{ bounded }, \quad 0 \le t \le T, \tag{1.3}$$

where r is the radial coordinate. For simplicity, in this paper we consider the initial temperature vanish. If the initial condition $u(r,0) = \varphi(r)$ not vanish, the problem can be divided into a well-posed problem and an ill-posed problem with $u(r,0) = 0$.

Getting here, an inverse heat source problem is formulated that is problem (1.1)–(1.3), which is to recover the pair of functions (u,f). This problem is ill-posed problem (The details can be seen in Section 2). Hence, a regularization is needed. But to the author's knowledge, so far there is no regularization theory with error estimate for problem (1.1)–(1.3).

Carasso [21], Fu [22] and Cheng et al. [23,24] applied a simplified Tikhonov regularization method to approximate inverse heat conduction problem and given the stability theory with explicitly error estimate, respectively. In this paper, we will use a simplified Tikhonov regularization method to solve problem (1.1)–(1.3).

This paper is organized as follows: in Section 2, the formulation of solution of problem (1.1)–(1.3) is given. In Section 3, a simplified Tikhonov regularization method with a logarithmic type error estimate is provided.

2. Formulation of solution of problem (1.1)–(1.3)

Throughout this paper, we denote by $L^2[0,r_0;r]$ the Hilbert space of Lebesgue measurable functions f with weight r on $[0,r_0]$. $(\cdot,\cdot)$ and $\|\cdot\|$ denote inner and norm on $L^2[0,r_0;r]$, respectively, with the norm

$$\|f\| = \left(\int_0^{r_0} r|f(r)|^2 \mathrm{d}r\right)^{1/2}.$$

As a solution of problem (1.1)–(1.3) we understand a function $u(r,t)$ satisfying (1.1)–(1.3) in the classical sense and for every fixed $r \in [0,r_0]$, the function $u(r,\cdot) \in L^2[0,T]$. In this class of functions, if the solution of problem (1.1)–(1.3) exists, then it must be unique [7]. We assume $u(r,t)$ is the unique solution of problem (1.1)–(1.3). Applying the method of separation of variable, we have the following lemma.

Lemma 2.1. *If the solution of problem (1.1)–(1.3) exists, then it is given by*

$$u(r,t) = \sum_{n=1}^{\infty} f_n \left(\int_0^t \mathrm{e}^{-\left(\frac{\mu_n}{r_0}\right)^2 (t-\tau)} \mathrm{d}\tau\right) J_0\left(\frac{\mu_n r}{r_0}\right), \tag{2.1}$$

and

$$f_n = \frac{2\left(\int_0^T \mathrm{e}^{-\left(\frac{\mu_n}{r_0}\right)^2 (T-\tau)} \mathrm{d}\tau\right)^{-1}}{r_0^2 J_1^2(\mu_n)} \int_0^{r_0} r g(r) J_0\left(\frac{\mu_n r}{r_0}\right) \mathrm{d}r, \quad n = 1, 2, \ldots, \tag{2.2}$$

where $J_0(z)$ and $J_1(z)$ denote 0th order and 1st order Bessel function, respectively [25], *$\{\mu_n\}_{n=1}^{\infty}$ are the sequence of root of equation $J_0(z) = 0$ and satisfy*

$$0 < \mu_1 < \mu_2 < \cdots < \mu_n < \cdots, \quad \lim_{n\to\infty} \mu_n = \infty.$$

Proof. Applying the method of separation of variable, we seek a solution of problem (1.1)–(1.3) with the form

$$u(r,t) = \upsilon(t)R(r). \tag{2.3}$$

Substitution of (2.1) into Eq. (1.1) and boundary conditions (1.3), we discover that $R(r)$ satisfies the following equation and boundary conditions:

$$R''(r) + \frac{1}{r}R'(r) + \lambda R(r) = 0, \quad 0 < r < r_0, \tag{2.4}$$

$$R(r_0) = 0, \tag{2.5}$$

$$|R(0)| < +\infty, \tag{2.6}$$

where λ are unknown constants. According to Jiang et al. [26], the eigenvalues of problem (2.4)–(2.6) are

$$\lambda_n = \left(\frac{\mu_n}{r_0}\right)^2, \quad n = 1, 2, \ldots,$$

and the corresponding eigenfunctions are

$$R_n(r) = J_0\left(\frac{\mu_n r}{r_0}\right), \quad n = 1, 2, \ldots,$$

where μ_n satisfy equation

$$J_0(z) = 0$$

and

$$0 < \mu_1 < \mu_2 < \cdots < \mu_n < \cdots, \quad \lim_{n\to\infty} \mu_n = \infty.$$

Thus the solution $u(r, t)$ and heat source term $f(r)$ of problem (1.1)–(1.3) can be represented as follows

$$u(r, t) = \sum_{n=1}^{\infty} \upsilon_n(t) R_n(r), \tag{2.7}$$

$$f(r) = \sum_{n=1}^{\infty} f_n R_n(r), \tag{2.8}$$

and

$$f_n = \frac{\int_0^{r_0} r f(r) J_0\left(\frac{\mu_n r}{r_0}\right) \mathrm{d}r}{\int_0^{r_0} r J_0^2(\frac{\mu_n r}{r_0}) \mathrm{d}r} = \frac{2}{r_0^2 J_1^2(\mu_n)} \int_0^{r_0} r f(r) J_0\left(\frac{\mu_n r}{r_0}\right) \mathrm{d}r, \quad n = 1, 2, \ldots.$$

According to the properties of 0th order Bessel function $J_0(x)$, the eigenfunction system $J_0\left(\frac{\mu_n r}{r_0}\right)$ are complete, and orthogonal with weight r in $L^2[0, r_0; r]$ [26]. Substituting (2.7) and (2.8) into Eq. (1.1) with condition (1.2), we discover that $\upsilon_n(t)$ satisfies

$$\upsilon_n'(t) + \left(\frac{\mu_n r}{r_0}\right)^2 \upsilon_n(t) = f_n,$$
$$\upsilon_n(0) = 0.$$

Solving the above initial-value problem, there holds

$$\upsilon_n(t) = \int_0^t f_n \mathrm{e}^{-\left(\frac{\mu_n}{r_0}\right)^2 (t-\tau)} \mathrm{d}\tau, \quad n = 1, 2, \ldots.$$

Therefore, the corresponding solutions of Eqs. (1.1)–(1.3) are

$$u(r, t) = \sum_{n=1}^{\infty} f_n \left(\int_0^t \mathrm{e}^{-\left(\frac{\mu_n}{r_0}\right)^2 (t-\tau)} \mathrm{d}\tau\right) R_n(r),$$

and at $t = T$ there holds the expansions

$$g(r) = \sum_{n=1}^{\infty} f_n \left(\int_0^T \mathrm{e}^{-\left(\frac{\mu_n}{r_0}\right)^2 (T-\tau)} \mathrm{d}\tau\right) J_0\left(\frac{\mu_n r}{r_0}\right).$$

From the above formula we can obtain (2.2). This proves the lemma. $\square$

Utilizing differential mean-value theorem, then there exist positive constants t_n, $0 < t_n < T$ such that

$$g(r) = \sum_{n=1}^{\infty} f_n (T \mathrm{e}^{-\left(\frac{\mu_n}{r_0}\right)^2 (T-t_n)}) J_0\left(\frac{\mu_n r}{r_0}\right), \tag{2.9}$$

here t_n is a constant dependent of μ_n. Let

$$\omega_n(r) = \frac{\sqrt{2}}{r_0 J_1(\mu_n)} J_0\left(\frac{\mu_n r}{r_0}\right), \tag{2.10}$$

then the eigenfunction system

$$\omega_1(r), \omega_2(r), \dots, \omega_n(r), \dots$$

is an orthonormal system with weight r in $L^2[0, r_0; r]$.

Using (2.2) and (2.10), (2.9) can be rewritten as

$$g(r) = \sum_{n=1}^{\infty} (T e^{-\left(\frac{\mu_n}{r_0}\right)^2 (T-t_n)})(f(r), \omega_n(r))\omega_n(r). \tag{2.11}$$

From (2.11), we have

$$(g(r), \omega_n(r)) = (f(r), \omega_n(r))(T e^{-(\mu_n/r_0)^2 (T-t_n)}),$$

and

$$f(r) = \sum_{n=1}^{\infty} (1/T) e^{(\mu_n/r_0)^2 (T-t_n)} (g(r), \omega_n(r))\omega_n(r). \tag{2.12}$$

Since measurement errors exist in $g(r)$, the solution has to be reconstructed from noisy data $g_\delta(r)$ which is assumed to satisfy

$$\|g(\cdot) - g_\delta(\cdot)\| \le \delta, \tag{2.13}$$

here $g(\cdot)$ and $g_\delta(\cdot)$ belong to $L^2[0, r_0; r]$.

We assume also that there exists an a priori condition for problem (1.1)–(1.3):

$$\|f(\cdot)\|_p \le E, \quad p > 0, \tag{2.14}$$

where $\|f(\cdot)\|_p$ is defined by

$$\|f(\cdot)\|_p = \left\| \sum_{n=1}^{\infty} (1+n^2)^{\frac{p}{2}} (f(\cdot), \omega_n(\cdot))\omega_n(\cdot) \right\|.$$

Applying (2.12) yields

$$\|f(\cdot)\| = \left(\sum_{n=1}^{\infty} (1/T)^2 e^{2\left(\frac{\mu_n}{r_0}\right)^2 (T-t_n)} |(g(r), \omega_n(r))|^2 \right)^{1/2}. \tag{2.15}$$

Note that $\mu_n \to +\infty$ tends to infinity as n tends to infinity. Then we have $(1/T)e^{\left(\frac{\mu_n}{r_0}\right)^2 (T-t_n)}$ tends to infinity as n tends to infinity. Combining with the boundedness of function $J_0\left(\frac{\mu_n r}{r_0}\right)$, formula (2.15) implies a rapid decay of $|(g(r), \omega_n(r))|$ for n large enough. But such a decay is not likely to occur in the measured noisy data $g_\delta(r)$ at $t = T$. So, small perturbation of $g(r)$ can blow up and completely destroy the solution $f(r)$ i.e., problem (1.1)–(1.3) is ill-posed.

3. Regularization and error estimate

We define an operator $K : f(\cdot) \to g(\cdot)$, then problem (1.1)–(1.3) in the interval $0 \le r \le r_0$ can be rewritten as the following operator equation:

$$Kf(r) = g(r), \quad 0 \le r \le r_0. \tag{3.1}$$

Using formula (2.11), there holds

$$Kf(r) = \sum_{n=1}^{\infty} k_n (f(r), \omega_n(r))\omega_n(r). \tag{3.2}$$

Consequently, K is linear self-adjoint compact operator with eigenvalues

$$k_n = T e^{-(\mu_n/r_0)^2 (T-t_n)} \tag{3.3}$$

and eigenelements ω_n.

Then for disturbed data $g_\delta(t)$, the Tikhonov regularization can be used, i.e., we seek a function f_α^δ which minimizes the quantity

$$J_\alpha(f^\delta) := \|Kf^\delta - g_\delta\|^2 + \alpha^2 \|f^\delta\|^2. \tag{3.4}$$

We shall prove the following lemma.

Lemma 3.1. *Let $f_\alpha^\delta \in L^2[0, r_0; r]$, then there exists a unique solution to the above minimization problem. It is given by*

$$f_\alpha^\delta(r) = \sum_{n=1}^{\infty} \frac{(1/T)e^{(\mu_n/r_0)^2(T-t_n)}}{1 + (\alpha/T)^2 e^{2(\mu_n/r_0)^2(T-t_n)}} (g_\delta, \omega_n)\omega_n. \tag{3.5}$$

Proof. Let I denote the identity operator in $L^2[0, r_0; r]$ and K^* be the adjoint of K. Then, by the Theorem 2.11 in [27], the Tikhonov functional J_α given by (3.4) has a unique minimum $f_\alpha^\delta \in L^2[0, r_0; r]$, and f_α^δ is the unique solution of the normal equation

$$K^*Kf_\alpha^\delta + \alpha^2 f_\alpha^\delta = K^* g_\delta, \quad \alpha > 0. \tag{3.6}$$

Since K is a self-adjoint operator, i.e., $K = K^*$, combining (3.2) with (3.6) we have

$$\sum_{n=1}^{\infty}(k_n^2 + \alpha^2)(f_\alpha^\delta, \omega_n)\omega_n = \sum_{n=1}^{\infty} k_n(g_\delta, \omega_n)\omega_n,$$

thus

$$(k_n^2 + \alpha^2)(f_\alpha^\delta, \omega_n) = k_n(g_\delta, \omega_n),$$

and

$$(f_\alpha^\delta, \omega_n) = \frac{k_n}{(k_n^2 + \alpha^2)}(g_\delta, \omega_n).$$

Therefore there holds

$$f_\alpha^\delta(r) = \sum_{n=1}^{\infty}(f_\alpha^\delta, \omega_n)\omega_n = \sum_{n=1}^{\infty} \frac{k_n^{-1}}{1 + \alpha^2 k_n^{-2}}(g_\delta, \omega_n)\omega_n.$$

The lemma is proved. □

We call $f_\alpha^\delta(r)$ given by (3.5) the Tikhonov approximation of the exact solution $f(r)$ given by (2.12) of problem (1.1)–(1.3) in the interval $0 \le r \le r_0$. It is interesting to compare formula (2.12) for the exact solution $f(r)$ with formula (3.5) for its Tikhonov approximation $f_\alpha^\delta(r)$. Clearly, the regularization procedure consists in replacing the unknown $g(r)$ with an appropriately filtered noisy data $g_\delta(r)$. The filter in (3.5) attenuates the high frequencies in $g_\delta(r)$ in a manner consistent with the goal of minimizing the quantity (3.4). By this idea we can replace the filter $\frac{(1/T)e^{(\mu_n/r_0)^2(T-t_n)}}{1+(\alpha/T)^2e^{2(\mu_n/r_0)^2(T-t_n)}}$ with another filter $\frac{(1/T)e^{(\mu_n/r_0)^2(T-t_n)}}{1+(\alpha/T)^2e^{2(\mu_n/r_0)^2T}}$ and introduce a new approximation $f_{\alpha,*}^\delta(r)$ of the solution $f(r)$ of problem (1.1)–(1.3) in the interval $0 \le r \le r_0$:

$$f_{\alpha,*}^\delta(r) = \sum_{n=1}^{\infty} \frac{(1/T)e^{(\mu_n/r_0)^2(T-t_n)}}{1 + (\alpha/T)^2 e^{2(\mu_n/r_0)^2T}} (g_\delta, \omega_n)\omega_n. \tag{3.7}$$

We call $f_{\alpha,*}^\delta(r)$ above the simplified Tikhonov approximations of the solution $f(r)$ of problem (1.1)–(1.3) for $0 \le r \le r_0$.

According to the properties of 0th order Bessel function $J_0(x)$, there exists a positive constant c such that, for all natural number n

$$n \ge c\mu_n. \tag{3.8}$$

In order to obtain stability estimate for the regularized solution, we need the following lemma and its proof is similar to that of lemma 3.2 in [28].

Lemma 3.2. *Let $T > 0, \alpha > 0$, then*

$$\sup_{s>0} \frac{e^{(T/r_0^2)s}}{1 + (\alpha/T)^2 e^{(2T/r_0^2)s}} \le \frac{T}{\alpha}. \tag{3.9}$$

Moreover, if $0 < \alpha < 1/\sqrt{3}$, $p > 0$, $T(p+1)/p \ge 1$, then there holds

$$\sup_{s>0} \frac{e^{(2T/r_0^2)s}(1+s)^{-p/2}}{1 + (\alpha/T)^2 e^{(2T/r_0^2)s}} \le \left(\frac{T}{\alpha}\right)^2 \left(\frac{r_0^2}{T} \ln \frac{1}{\sqrt{3}\alpha}\right)^{-\frac{p}{p+1}}. \tag{3.10}$$

Theorem 3.3. *Let $f(r)$ given by (2.12) be the exact heat source history for $r \in [0, r_0]$ and $f_\alpha^\delta(r)$ given by (3.7) be the regularized approximation heat source to $f(r)$. Let the measured data at $t = T$, $g_\delta(r)$, satisfy the condition (2.13) and a priori condition (2.14) be valid. If we select the regularization parameter α as*

$$\alpha = \frac{\delta}{E}\left(\ln\frac{E}{\delta}\right)^{\frac{p}{p+1}} \tag{3.11}$$

then there holds the stability estimate

$$\|f(\cdot) - f^{\delta}_{\alpha,*}(\cdot)\| \le E\left(\ln\frac{E}{\delta}\right)^{-\frac{p}{p+1}} \left(c_1(T/r_0^2)^{\frac{p}{p+1}} + 1 + o(1)\right) \quad \text{for } \delta \to 0, \tag{3.12}$$

where $c_1 = (\min\{1, c^2\})^{-p/2}$.

Proof. Due to (2.12), (3.7) and (3.8) there holds

$$\begin{aligned}
\|f(\cdot) - f^{\delta}_{\alpha,*}(\cdot)\| &= \left\| \sum_{n=1}^{\infty} k_n^{-1}(g,\omega_n)\omega_n - \sum_{n=1}^{\infty} \frac{k_n^{-1}}{1+(\alpha/T)^2 e^{2\left(\frac{\mu_n}{r_0}\right)^2 T}}(g_\delta,\omega_n)\omega_n \right\| \\
&\le \left\| \sum_{n=1}^{\infty} \frac{k_n^{-1}(\alpha/T)^2 e^{2\left(\frac{\mu_n}{r_0}\right)^2 T}}{1+(\alpha/T)^2 e^{2\left(\frac{\mu_n}{r_0}\right)^2 T}}(g,\omega_n)\omega_n \right\| + \left\| \sum_{n=1}^{\infty} \frac{k_n^{-1}}{1+(\alpha/T)^2 e^{2\left(\frac{\mu_n}{r_0}\right)^2 T}}(g-g_\delta,\omega_n)\omega_n \right\| \\
&\le \sup_{\mu_n>0} \frac{(\alpha/T)^2 e^{2\left(\frac{\mu_n}{r_0}\right)^2 T}(1+n^2)^{-p/2}}{1+(\alpha/T)^2 e^{2\left(\frac{\mu_n}{r_0}\right)^2 T}} \left\| \sum_{n=1}^{\infty}(1+n^2)^{p/2}k_n^{-1}(g,\omega_n)\omega_n \right\| \\
&\quad + \sup_{\mu_n>0} \frac{(1/T)e^{\left(\frac{\mu_n}{r_0}\right)^2 (T-t_n)}}{1+(\alpha/T)^2 e^{2\left(\frac{\mu_n}{r_0}\right)^2 T}} \left\| \sum_{n=1}^{\infty}(g-g_\delta,\omega_n)\omega_n \right\| \\
&\le \sup_{\mu_n>0} \frac{(\alpha/T)^2 e^{2\left(\frac{\mu_n}{r_0}\right)^2 T} c_1(1+\mu_n^2)^{-p/2}}{1+(\alpha/T)^2 e^{2\left(\frac{\mu_n}{r_0}\right)^2 T}} \|f\|_p + \sup_{\mu_n>0} \frac{(1/T)e^{\left(\frac{\mu_n}{r_0}\right)^2 T}}{1+(\alpha/T)^2 e^{2\left(\frac{\mu_n}{r_0}\right)^2 T}} \|g-g_\delta\|,
\end{aligned}$$

here $c_1 = (\min\{1, c^2\})^{-p/2}$. Let $s = \mu_n^2$, combining with conditions (2.13) and (2.14) and inequalities (3.9), (3.10), we obtain

$$\begin{aligned}
\|f(\cdot) - f^{\delta}_{\alpha,*}(\cdot)\| &\le c_1 E(\alpha/T)^2 \sup_{s>0} \frac{e^{(2T/r_0^2)s}(1+s)^{-p/2}}{1+(\alpha/T)^2 e^{(2T/r_0^2)s}} + \frac{\delta}{T}\sup_{s>0}\frac{e^{(T/r_0^2)s}}{1+(\alpha/T)^2 e^{(2T/r_0^2)s}} \\
&\le c_1 E\left(\frac{r_0^2}{T}\ln\frac{1}{\sqrt{3}\alpha}\right)^{-\frac{p}{p+1}} + \delta\alpha^{-1}.
\end{aligned}$$

From the choice of α given by (3.11), we have

$$\|f(\cdot) - f^{\delta}_{\alpha,*}(\cdot)\| \le E\left(\ln(E/\delta)\right)^{-\frac{p}{p+1}} \left[c_1 \left(\frac{(T/r_0^2)\ln(E/\delta)}{\ln(E/(\sqrt{3}\delta)) - \frac{p}{p+1}\ln(\ln(E/\delta))} \right)^{\frac{p}{p+1}} + 1 \right].$$

Note that, for $\delta \to 0$

$$\frac{\ln(E/\delta)}{\ln(E/(\sqrt{3}\delta)) - \frac{p}{p+1}\ln(\ln(E/\delta))} \to 1.$$

Thus estimate (3.12) is proved. □

We can see that estimate (3.12) is a logarithmical stability estimate which is similar to the convergence estimate in [29], and [28] on the internal surface $r = r_0$.

Remark 3.4. In general, the a priori bound E is unknown in practice, in this case, for Theorem 3.3, with

$$\alpha = \delta\left(\ln\frac{1}{\delta}\right)^{\frac{p}{p+1}}$$

then there holds the stability estimate

$$\|f(\cdot) - f^{\delta}_{\alpha,*}(\cdot)\| \le \left(\ln\frac{1}{\delta}\right)^{\frac{p}{p+1}} \left(c_1 E(T/r_0^2)^{\frac{p}{p+1}} + 1 + o(1)\right) \quad \text{for } \delta \to 0.$$

Acknowledgments

The authors would like to express their gratitude to the reviewers for their valuable comments and suggestions.

References

[1] J.R. Cannon, Determination of an unknown heat source from overspecified boundary data, SIAM J. Numer. Anew. 5 (1968) 275–286.
[2] J.R. Cannon, S.P. Esteva, An inverse problem for the heat equation, Inverse Problems 2 (1986) 395–403.
[3] P. Duchateau, W. Rundell, Unicity in an inverse problem for an unknown reaction term in a reation-diffusion-equation, J. Differential Equations 59 (1985) 155–164.
[4] J.R. Cannon, P. Duchateau, Structual identification of an unknown source term in a heat equation, Inverse Problems 14 (1998) 535–551.
[5] A.J. Silva Neto, M.N. Özisik, Two-dimensional inverse heat conduction problem of estimating the time-varying strength of a line heat source, J. Appl. Phys. 71 (1992) 5357–5362.
[6] F. Hettlich, W. Rundell, Identification of a discontinous source in the heat equation, Inverse Problems 17 (2001) 1465–1482.
[7] J. Hadamard, Lectures on Cauchy Problem in Linear Partial Differential Equations, Oxford University Press, London, 1923.
[8] J.R. Cannon, S.P. Esteva, Uniqueness and stability of 3D heat source, Inverse Problems 7 (1991) 57–62.
[9] M. Choulli, M. Yarmamoto, Conditional stability in determining a heat source, J. Inverse Ill-posed Probl. 12 (3) (2004) 233–243.
[10] Z. Yi, D.A. Murio, Source term identification in 1-D IHCP, Comput. Math. Appl. 47 (2004) 1921–1933.
[11] Z. Yi, D.A. Murio, Source term identification in 2-D IHCP, Comput. Math. Appl. 47 (2004) 1517–1533.
[12] A. Farcas, D. Lesnic, The boundary-element method for the determination of a heat source dependent on one variable, J. Eng. Math. 54 (2006) 247–253.
[13] A.G. Fatullayev, Numerical solution of the inverse problem of determining an unknown source term in a heat equation, Math. Comput. Simulation 8 (2) (2002) 161–168.
[14] M. Dehghan, M. Tatari, Determination of a contral parameter in a one-dimensional parabolic equation using the method of radial basis functions, Math. Comput. Modelling 8 (2) (2002) 161–168.
[15] P.S. Mahar, B. Datta, Optimal identification of ground-water pollution source and parameter estimate, J. Water. Res. Plan. Manag. 127 (1) (2001) 20–29.
[16] D. Constales, J. Kacur, K.R. Van, Parameter identification by a single injection-extraction well, Inverse Problems 18 (2002) 1605–1620.
[17] G.S. li, Y.C. Ma, K.T. Li, An inverse parabolic problem for nonlinear source term with nonlinear boundary condition, J. Inverse Ill-posed Probl. 11 (4) (2003) 371–387.
[18] G.S. Li, Y.G. Tan, A conditional stability for an inverse problem arising in groundwater pollution, J. Chin. Univ. 14 (3) (2005) 217–225.
[19] D.D. Trong, N.T. Long, P.N.D. Alain, Nonhomogeneous heat equation: Identification and regularization for the inhomogeneous term, J. Math. Anal. Appl. 312 (2005) 93–104.
[20] Y. Liu, Cylinder Functions, Industry of National Defence Press, 1983 (in Chinese).
[21] A. Carasso, Determining surface temperature from interior observations, SIAM J. Appl. Math. 42 (1982) 558–574.
[22] C.-L. Fu, Simplified Tikhonov and Fourier regularization methods On a general sideways parabolic equation, J. Comput. Appl. Math. 167 (2004) 449–463.
[23] W. Cheng, C.L. Fu, Z. Qian, Two regularization methods for a spherically symmetric inverse heat conduction problem, Appl. Math. Modelling 32 (4) (2008) 432–442.
[24] W. Cheng, C.L. Fu, Two regularization methods for an axisymmetric inverse heat conduction problem, J. Inverse Ill-posed Probl. 17 (2009) 157–170.
[25] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, Dover Publications, Inc., New York, 1972.
[26] L.S. Jiang, Y.J. Chen, X.H. Liu, F.K. Yi, Teaching Materials to Equation of Mathematical Physics, Higher Education Press, 1995 (in Chinese).
[27] A. Kirsch, An Introduction to the Mathematical Theory of Inverse Problems, Springer, New York, 1996.
[28] W. Cheng, C.L. Fu, Z. Qian, A modified Tikhonov regularization method for a spherically symmetric three-dimensional inverse heat conduction problem, Math. Comput. Simulation 75 (3–4) (2007) 97–112.
[29] M.T. Nair, U. Tautenhahn, Lavrentiev regularization for linear ill-posed problems under general source conditions, J. Anal. Appl. 23 (1) (2004) 167–185.